The devil in details: Mathematics teaching and learning as managing inter-discursive gaps

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Once teaching-learning events are conceptualised as inter-discursive encounters, it becomes clear that mathematics classroom talk is rife with invisible pitfalls. There are many types of unacknowledged discursive gaps, some of them necessary for learning, and some potentially harmful. Such gaps may exist also between the teacher's intentions and her own habitual moves, most of which are too brief and automatic to be controlled. Unknown to the teacher, her basic communicational routines may constitute invisible crevices through which the prejudice enters the conversation on mathematical objects. In this talk, I argue that if the devil is in the finest detail of classroom communication, it is the detail that must be considered in the attempts to exorcise the devil. I begin with illustrations of these claims and conclude with a reflection on how mathematics teachers may sensitise themselves to discursive pitfalls, how they and their students can benefit from those communicational gaps that are likely to generate learning, and how they can cope with those divides that hinder the process or infect it with unwanted messages.

Humans, unlike most other species, can exist only as a part of a society. But while our very survival may depend on effective interpersonal exchanges, our communication is only too prone to failure. Some go so far as to claim that within this context, failure constitutes the default option, whereas success should be regarded as almost a miracle (Reddy, 1979).

Perhaps the most challenging aspect of communicational breakdowns is that they often go unnoticed. Paraphrasing Hamlet, one can say that there are more communicational pitfalls in heaven and earth than are dreamt of by philosophers or suspected by ordinary people. These pitfalls tend to hide in unnoticeable details of interlocutors' actions. Obviously, people trying to reach one another across a hidden communicational gap risk falling to the bottom. As blind to the fall as they were to the pitfall, they are likely to leave the exchange with unhelpful interpretations of each other's intentions. At home, it may hurt their relationships; in the classroom, it may stymie their learning. In the words of George Bertrand Shaw, "The single biggest problem in communication is the illusion that it has taken place". This paper is about guarding ourselves against this illusion by becoming alert to communicational pitfalls.

Some may claim that the existence of certain communicational gaps is inherent to learning and thus little can be done against them. Yet, I wish to argue that even when a gap is necessary for the further development of mathematical discourse, the importance of our awareness to its existence cannot be overstated. Indeed, exposing the gaps is a critical step in turning them from obstacles into opportunities for learning. Clearly, being constantly on the watch for hidden communicational hurdles will also help in guarding ourselves against the adverse impact of those gaps that could be avoided.

In what follows, I illustrate the claim about the omnipresence of communication gaps with examples from mathematics classrooms. With the help of specially designed conceptual apparatus, evolving around the vision of learning as a process of routinisation of our actions, I zoom into the data and identify seemingly negligible details that may constitute, for better or worse, powerful shapers of students' learning.

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Communicational gaps

In the classrooms, the presence of invisible communication pitfall may signal itself by puzzling occurrences, for which neither the teacher nor an external observer can provide an immediate explanation. The danger of the *illusion* of communication, however, is at its worst when nothing seems unusual and the communicational glitch, although quite real, does not manifest itself in a palpable way.

Consider, for example, the exchange between a teacher and her student, presented in Table 1. What happens in this brief episode is so familiar that the claim about the student's initial difficulty as due to any communicational issue is likely to be met with scepticism. Indeed, nothing seems surprising that the child who is evidently quite new to the topic of fractions has difficulty multiplying a fraction by a whole number. It is also not startling that after the teacher's additional probing (see turns [3] and [5]) and with some effort on the part of the student, the proper answer is finally produced ([6]). The teacher summarised saying that a bit of effort was all the boy needed to succeed ([7]). In making this statement, she implied that the learner was already acquainted with the necessary procedure, but was not yet quite proficient in its application and performance.

Table 1
Example I: Multiplying by Fraction

#	Speaker	What is said	What is done
1	Teacher:	So, what is?	Writes $\frac{1}{3} \cdot 12$
2	Student:	......	
3	Teacher:	Try again, one third times twelve	
4	Student:	I think.... Don't know...	
5	Teacher:	Once again, one third of twelve	
6	Student:	Ahm..... It's four	
7	Teacher:	Great. See, when you think about it, you know how to do it!	

As unproblematic as this simple account seems to be, at a closer look it leaves an important question unanswered. Yes, the child did seem to make an effort. Yet, although he clearly tried hard already the first time round, he was able to produce an answer only after the teacher's third attempt. What was it about this third question ([5]) that brought the sudden insight? How was this query different from the previous ones ([1], [3])? Some scrutiny of the three instances may suffice to realise that each of the three utterances referred to the required operation in its own way:

1. with the help of the written expression '$^1/_3 \cdot 12$' ([1])
2. orally, with the expression "one third times twelve" ([3])
3. orally, with "one third of twelve" ([5]).

The first two of these renditions that make use of distinctly mathematical symbol '·' and the word "times", that belong to the formal discourse on numbers. The last utterance, which speaks about "one third *of* twelve", may be a part of the child's everyday talk and can belong to the repertoire also of a person with no access to formal mathematics. The first two utterances directed the child to as-yet unfamiliar numerical operation, whereas the third one required the everyday action of identifying a familiar part of a whole.

The difference between this account and the one offered by the teacher is subtle, may even appear negligible, but it is highly consequential. With her vision of the current state of the student's learning, the teacher will likely emphasise the need for fostering the child's procedural proficiency. In contrast, the realisation that the student might had participated in a discourse different from her own and that, in result, the task he tried to perform was not the one she had in mind will turn her attention to the conceptual side of the story. Building on the resulting conceptual interpretation, she may decide to focus on helping the learner to see connections between his everyday talk and the mathematical discourse of multiplication.

In this analysis, I exemplified the way in which we can make ourselves aware of subtle communicational issues that, if unrecognised, may lead the teacher to unhelpful pedagogical decisions, but if noticed, are likely to give rise to opportunities for significant learning. The terms such as 'discourse' or 'task' have been used in this analysis freely, without a proper introduction. The next section provides what is missing. After defining the terms as they are to be understood within a discursive theory of learning, I will be able to operationalise the notion of communicational gap and instantiate ways in which the risks of such gap can be significantly reduced and its potential as an opportunity for learning considerably increased.

Operationalising the construct of communicational gap

Mathematics as discourse

In this paper, the word *discourse* is used as referring to the special form of communication, characteristic of a particular community. The community may be that of scientists, chess players or of art theorists. Most relevantly for our present context, it may be a community of mathematicians or of mathematics classrooms. Whereas each such community is unified by its members common interest, activity or cultural practice, its discourse is designed specifically to tell stories with which this activity or practice can be usefully mediated.

Thus, the first characteristic of a discourse that sets this discourse apart from any other is its collection of *endorsed narratives* about this discourse's focal objects. The adjective 'endorsed' indicates that these narratives are considered by its participants as faithful accounts of the state of affairs in the world and thus, as reliable guides for future actions. In mathematics, endorsed narratives are about such abstract objects as numbers, sets, geometric figures, functions, etc. The communicational tools with the help of which these stories are forged and substantiated constitute additional set of characteristics that make the discourse distinguishable from other ones. Thus, there is the set of special-purpose *keywords* pertaining to the focal objects and actions of the discourse. In mathematics, these are words such as 'number', 'function', 'triangle', 'adding', 'differentiating', etc. Although many of these words may be known also from everyday talk, in specialised discourse their use is different and defined more strictly. Another special feature of a discourse is the set of special *visual mediators* that help in ensuring the effectiveness of communication. Algebraic symbols and graphs are among the most useful mediators of mathematical discourse. Finally, discourses are made distinct by their *routines,* the recurrent ways of performing different kinds of tasks, such as, in mathematics, calculating, proving or performing geometric constructions with the help of ruler and compass. Some of the routines are algorithmic, some are more of a 'rule of a thumb'. This last characteristic, routine, being particularly relevant to the topic of communicational gaps, requires some elaboration.

More about routines

Routine, far from being just an optional way of acting (and a rather boring one, some may say, because of its repetitive nature), is what makes us able to act in the first place. Indeed, it is thanks to routines that we know how to act whenever we feel expected to do something, which is most of the time. In such situation, to react to the prompt in an immediate way, the best we can do is to turn to those familiar ways of acting that worked for us in the past in a similar situation (or what we consider as such). This, indeed, was what the student in Example II was able to do when he eventually found the way to answer the interviewer's question: he recalled what was done when somebody asked the question of the form "What is one third of X?", with X being a set of a certain size (12 items, in this case).

To operationalise the construct of routine, there is a need for some auxiliary notions. Thus, the situation in which a person feel she is obliged to act will be called *task-situation*. Such situation may arise of itself, as is the case when one feels cold or hungry. Task-situation may also be created by asking questions. In Example II, this is what the teacher did three times, in turns [1], [3] and [5]. Once a person finds herself in a task-situation, she needs to decide about her *task*, that is, about what needs to be done, and about a *procedure* that suits that task. Deliberately or instinctively, this person will probably try to do this by recalling precedents. *Precedent* is any previous task-situations that appears to a person as sufficiently similar to the present one to justify doing now what was done then. Given suitable precedents, she will see it as her task to act in such a way as to ensure the reoccurrence of specific aspects of the precedent task-situation. For instance, while feeling hungry, she will probably see it as her task to make the sense of hunger disappear. Her procedure will be the prescription for action that, according to her interpretation, guided the previous task performer. In hunger instigated task-situation, the procedure may be a walk to a fridge and helping herself to some food.

Once the search for task and procedure is successfully completed, the person is ready to act. Note that in most daily task-situations, especially in those with which we are intimately familiar, this initial step is intuitive rather than conscious and deliberate, and rarely makes us slow down for reflection. We may say that the *task-procedure pair* resulting from one's search, being a prescription for an emerging pattern, is this person's *routine* for dealing with the given task-situation. Learning can now be seen as a process of routinisation of our action (Lavie et al. 2019).

Discursive gaps and their sources

In the light of the above definition, routine is not a free-floating, context-free phenomenon. I will now argue that routines depend on task-situations and on their interpreters. To put it differently, different people may interpret the same task-situation in different ways, ending up with different tasks, to be performed with the help of different procedures. To show this, I need to take a closer look at how people decide about tasks and procedures.

On the face of it, the search for routines that would fit particular task-situations appears so demanding, it is more likely to fail than succeed. Indeed, we would have little chance to succeed in interpreting task-situations if we were to search precedents among *all* past events, from all times and all locations. Fortunately, search spaces tend to shrink considerably the moment we enter a specific task-situation. Imperceptibly to ourselves, we react to such a situation with a choice of a discourse in which to think about this situation. The subsequent search for precedents will be restricted to past situations in which people had recourse to this discourse. With different discourses come different routines, that is, different ways of acting. Thus, more often than not, a task-situation created by the mathematics teacher automatically directs the students to the discourse of this teacher's classroom, and to routines that were employed there, preferably in the most recent past. And vice versa: task-situation created in out-of-school context is likely to direct potential performers to everyday discourse, barring them from any other. Indeed, we tend to close ourselves in discourses we associate with a given situation and this tendency may account for the phenomenon known as *situativity* of learning (Brown et al., 1989; Lave, 1988), that is, for the fact that most people do not usually apply in one context routines they have learned in another. In particular, this maybe the reason why mathematics learned in school is, in most cases, practically absent from our daily lives.

It is this tendency for associating situations with discourses that may be responsible for the event presented in Table 1, in which the student reacted in different ways to what seemed to the teacher as mere repetitions of "the same" question. More generally, considering the dependence of our discursive choices on our past experience, it is only understandable that people participating in the same conversation would often turn to different discourses. In the next section, we use the former example, as well as some other ones, to show that the resulting communicational disparities carry both risk and promises, and that making them visible may help the teacher to turn the gaps from pitfalls into learning opportunities for her students.

Discursive gaps as opportunities for learning

The two examples to be presented in this section illustrate the thesis that discursive gaps, while constituting a treat to the process of learning, may also be indispensable for the development of mathematical discourse. In both these examples, a close analysis will show that two people engaged in a conversation with one another may, in fact, be participating in different discourses.

Example I: Opportunity for developing routines by bonding them with other ones

Back to the example presented in Table 1, I can now present the results of the former analysis with the help of the conceptual tools introduced above. Here is the new description: the three *task-situations* created by the teacher's questions [1], [3], and [5], although identical in the eyes of the teacher, were seen as different by the student. More specifically, questions [1] and [3] probably sent the child *searching for precedents* among past classroom situations in which a formal algorithm for multiplying fraction by a whole number was used. Question [5], on the other hand, might have brought to his mind everyday situations in which a conversation was about sharing a certain amount of cookies fairly between three friends. The *tasks* envisioned by the child as a result of these differing choices of discourses and precedents were also different: In the first case, he saw it as his job to perform the symbolic manipulation he learned in school. In the second case, his task was to find out what would be the share of one person if twelve items were distributed evenly between three people. This interpretation is summarised in Table 2.

An important insight about development of routines can be gained from this example. At a close look, these two tasks, as well as the resulting procedures, have little in common with one another. Yet, those who are well versed in multiplying by fractions and perform this operation almost automatically are usually oblivious to the difference between the sequences of actions required in these two cases. The long experience with the respective procedures might have blinded them to an interesting phenomenon that transpired very clearly from an ongoing PhD research on the development of the discourse on rational numbers[1]. Indeed, oldtimers to that discourse typically do not remember that they were probably well acquainted with words such as *half*, *quarter*, (*one*)-*third* or *three-quarters* well before they knew anything about the formal discourse on fractions. If so, they have also forgotten that once upon a time, these basic fraction words did not function for them as names of numbers, but were rather labels for some special routines. At that time, "finding a third of a pizza" meant not much more than a physical action of cutting the pizza into three parts, whereas "giving each of three children a third of the twelve cookies" meant the circular action of handing a single cookie to each of the children (usually while saying "one for you, and one for you…"), and repeating the action until none of the twelve cookies was left. At that time, the expression "$^1/_3 \cdot 12$" was meaningless. In other words, different rational numbers corresponded in the beginning to different procedures used in execution of different tasks. It took time until the different tasks consolidated into one, and the different procedures became alternative branches of a single algorithm.

Table 2
Discourses and routines in Example I

	The teacher's interpretation of the task-situations created by all her utterances	The student's interpretation of task-situation created by the teacher's utterance [5]
discourse	numerical	of parts and wholes
task	perform the formal numerical calculation $^1/_3 \cdot 12$	find one third of a set of 12
procedure	Apply the algorithm for multiplication of rational numbers	1. Divide the whole into the number of equal parts indicated by the name of the part 2. Take one part

As argued by Lavie et al. (2019), such *bonding*[2] of several routines and turning them into a single one constitutes one of the central mechanisms in the development of discourses. In the present case, many other routines that in the eyes of the beginner have little to do with the school discourse on fractions will yet be bonded with the formal operation "$^1/_3 \cdot 12$" before the full-fledged routine for multiplying rational numbers emerges. The process of gradual bonding will lead to successive extensions in the applications of the resulting super-routine known as multiplication of rational numbers. These developments will greatly increase the usefulness of the multiplication routine, and with it, that of the whole discourse of rational numbers.

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[1] The research, titled "Development of the discourse on rational numbers" is being conducted these days at the University of Haifa by Aya Steiner. Its partial results have been published in Steiner (2018).

[2] This type of bonding, one that happens between different procedures, is sometimes qualified with the adjectives *horizontal* or *external* so as to be distinguished from the bonding that occurs inside the procedure, and is thus known as *vertical* or *inner*.

Example II: Opportunity for meta-level learning

In this example, taken from a study on a 7th grade class learning about negative numbers (Sfard, 2007), a different type of discursive gap comes to the fore. Before explaining its nature and source, let us take a look at classroom events that signalled its existence.

At the time the event took place, the class has already discussed the multiplication of negative numbers by positive numbers, but some students were still questioning the claim that the result should be negative. The relevant episode began when the teacher declared that she was going to "explain" this fact in a new way. On this occasion, she would also show how the product of two negative numbers should be defined. As can be seen in the episode presented in Table 3, she decided to derive all this from the multiplication of natural numbers, with which the children were already well acquainted.

Table 3
Example II: Teacher demonstrates derives multiplication of integers

#	Speaker	What is said	What is done
1556a	Teacher:	Well, I wish to explain this now in a different way.	Points to [2 · (–3) = –6]
1556b			Writes on the blackboard the following column of equalities: 2 · 3 = 6 2 · 2 = 4 2 · 1 = 2 2 · 0 = 0 2 · (–1) = –2 2 · (–2) = –4 2 · (–3) = –6 While writing, she stops at each line and asks the children about the result before actually writing it down and stressing that the decrease of 1 in the multiplied number decreases the result by 2
1556c	Teacher:	Let us now compute (–2) times (–3) in a similar way.	As before, writes on the blackboard the following column of equalities, stopping at each line and asking the children about the result before actually writing it down and noting that the decrease of 1 in the multiplied number increases the result by 3; this rule, she says, must be preserved all along: 3 · (–3) = –9 2 · (–3) = –6 1 · (–3) = –3 0 · (–3) = 0 (–1) · (–3) = 3 (–2) · (–3) = 6

Table 4 shows objections raised by some students in reaction to the teacher's argument.

Table 4
Children's reactions to teacher's derivation of the laws of multiplication

#	Speaker	What is said
1557	Shai:	I don't understand why we need all this mess. Is there no simpler rule?
1559	Sophie:	And if they ask you, for example, how much is (–25) · (–3), will you start from zero, do 0 · (–3), and then keep going till you reach (–25) · (–3)?

The students seem to have misinterpreted the teacher's intentions. The teacher saw it as her task to justify the definition of integer multiplication by deriving it from operations on natural numbers[3]. In contrast, the children interpreted the teacher's performance as a presentation of a new algorithm for multiplication, which they then criticized as a rather cumbersome method for producing simple endorsed narratives such as (–2) · (–3) = 6 or (–25) · (–3) = 75. The nature of the resulting discursive gap is detailed in Table 5.

Table 5
Discourses and routines in Example II

	The teacher's interpretation of her own performance	Children's interpretation of the teacher's performance
discourse	of unsigned numbers	of integers
task	define "plus times minus"	calculate a product of a positive and negative number
procedure	build a list that leads form a known operation (multiplication of two natural numbers) to the desired ones ("plus times minus" and "minus times minus")	

Why this difference in the teacher's and students' interpretation of the task-situation? One explanation is that the children were still captive of the discourse of unsigned numbers. In that familiar discourse, numbers and numerical operations constituted a part of the external, mind-independent world. Indeed, so far, it was the world that dictated the result of all numerical operations, such as 2·3 or 5·½. In the discourse of signed numbers, in contrast, the nature of numeric operations seems to be established in the act of defining, as if by fiat. This change is tantamount to passing the power of deciding about what exists and what happens in mathematical universe from the external, natural powers – or maybe from the God – to humans. As such, it is difficult to accept, and even before that, to conceive.

Two discourses that differ in their routines for forging and endorsing narratives have been called *incommensurable* (Sfard 2007).[4] The transition from the discourse of natural

[3] Here, the set of natural numbers is regarded as including zero. The unspoken principle underlying the teacher's argument was that the definition of multiplication of integers would preserve some basic numerical laws that held in the realm of numbers so far.

[4] This difference means a change in meta-rules, that is, in the rules that govern the activity of mathematizing. Such meta-level change can lead to seemingly contradicting endorsements. And, indeed, the narrative "There is a number that is smaller than any other" that held in the discourse on natural numbers is one of the many that will have to be abandoned once this discourse is extended to the one on integers.

numbers to that of integers is one of several passages to an incommensurable discourse that the student will have to make in the process of learning. The learning that takes place during these passages has been described as *meta-level*, so as to signal that in this case, the learning involves not just an addition of new narratives, but also a change in how such narratives are created and endorsed. A successful meta-level learning closes the discursive gap that spurred this learning. This closure does not mean the disappearance of the former discourse – of the discourse of natural numbers in the present case. Rather, this old discourse is subsumed in the new one and subjected to its differing meta-rules.

Summary and conclusions: Implications for teaching

The two cases of discursive gap shown in this section shed much light on processes of discourse development. The first of them tells us something about the growth of routines: such growth involves turning a number of hitherto unrelated procedures into special cases of a single procedure for the execution of different variations of the same task. This means that task-situations seen by discursive oldtimers as "the same" (equivalent), may be seen by newcomers as different. The second example shows the inevitability of discursive gaps as those that spur learning (*meta-level* learning, in this case) in the first place. Indeed, every so often, further development of mathematical discourse will remain stymied until the students confront and overcome a discursive gap: until they face, and reconcile themselves with a discourse incommensurable with the one in which they participated so far. In sum, in both cases, the gap, far from being just a nuisance, is what spurs the development in the first place. As such, it is indispensable for learning.

Obviously, in cases such as those presented in this section, avoiding the gaps would preclude the possibility of learning. As such, it is not an option. Instead, one should try to minimize the risks of the gap and optimize its potential benefits. Yet, not only the students, but also teachers are rarely aware of discursive gaps such as those described in the two examples. It is by making them visible that the teacher may turn potential pitfalls into opportunities for learning. The question of how to do this must be left for another article.

Discursive gaps as a danger to teaching

Unlike in the case of discursive gaps that are necessary for students' learning and thus cannot be prevented, the two examples in this section show avoidable gaps that, if left unattended, are likely to distort teaching. In both cases, these gaps stem from the teacher's inadvertent participation in a discourse that clashes with her intentions.

Example III: Involuntary engagement in constructing students' identities

While in mathematics classroom, the students and the teachers are supposed to mathematize, that is, to participate in a discourse on mathematical objects. Yet, mathematical discourse, even when predominant, is rarely the only one. All along the mathematical conversation, participants also make statements about themselves and others. Although the *subjectifying* narratives (narratives about people, as opposed to those about mathematical objects) produced in the process may not be getting a direct attention, in a longer run that may have a considerable impact on the participants' identities, that is, on the stories they believe true about themselves and about others. When it comes to students' identities, particularly influential is the subjectifying activity of the teacher. Although in most cases the teacher would probably readily admit that she bears a major responsibility for how her students see themselves as learners, she may not be sufficiently alert to those aspects of her classroom performances that constitute the most powerful identity-builders. Indeed, as I will now show with the help of an example, the devil may hide in tiniest details of the teacher's actions. The most powerful may be those brief moves that the teacher performs automatically, without planning in advance, without explicitly monitoring them at the time of performance and without remembering afterwards.

The example that follows comes from a study devoted to middle school students' extracurricular mathematical activities organized and led by one of the researchers (Heyd-Metzuyanim & Sfard, 2012). In the case under consideration, a group of four students described by their regular mathematics teacher as "good" (having a history of above average achievement) attempted to solve a non-standard mathematical problem. After a brief period of individual grappling, the participant whom the researchers called Ziv declared that he had answered the question, and that he did it in more than one way. Encouraged by the instructor, the boy presented one of the solutions. Yet, although Ziv's account appeared to the researchers clear and helpful, it was rejected by his classmates as incomprehensible. Explanations by another student, Dan, who also claimed to have a solution, appeared confusing and inconclusive. In spite of this, the students who previously complained about "not understanding Ziv", listened to Dan carefully and later claimed to have benefitted from his account. This event left the instructor perplexed. She was not able to figure out the reason why the students refused to learn from a knowledgeable classmate, but were eager to seek help of the one who clearly experienced difficulties not much different from their own. At that day, she left the following note in her journal:

> Although nobody seemed to doubt the correctness of Ziv's solution, no visible effort was made to find out what his proposal was all about. Nothing indicated an interest in Ziv's explanation… On the other hand, the students seemed eager to learn from Dan, who himself was struggling for understanding, and who offered ideas that seemed too blurred to be truly helpful… Unimpressed by [Ziv's] solution .... the students let the obvious opportunity for learning slip away."

It was only in later analyses that the researchers were able to account for what happened. While scrutinizing the classroom talk, they noticed a feature of which they were previously unaware: an undercurrent of intensive subjectifying was going on within what might appear to be just a regular mathematical conversation. If we remained unaware of this fact, it was because subjectifying utterances, when interjected into strenuous mathematical debates, tend to be ignored. If we were able to do some work on them now, it was because prior to the analysis, we systematically extracted them from their context and collected them together in a single table. Here, they were segregated according to their authors and to the persons about whom they spoke.

The result was startling. The majority of subjectifying utterances turned out to be about Ziv. Whether addressed to him or to another group member, whether made by himself or by another participant, these utterances were evidently evoked by the teacher's decisions and moves. Indeed, acting as the conversation coordinator, she never missed an opportunity to show her confidence in Ziv's ability to enlighten his classmates. The teacher expressed this belief in many different ways: by repeatedly urging Ziv to present his solutions ("Until now, you haven't told us what you have understood from this question" [266]), by exhorting others students to listen ("Dan listen to Ziv now" [383]), and by explicitly assessing Ziv's superior ability to understand the problem ("[Y]ou're the only one who understood [the question]"[99]). Through these and similar subjectifying actions the teacher, imperceptibly to herself, was gradually building Ziv's identity as mathematically versed and as the discourse leader. In an indirect way, these subjectifying moves identified the rest of the group as somehow inferior. Not surprisingly, Ziv's classmate reacted hostilely, trying to deny the power evidently ceded to Ziv by the instructor. Beginning with angry claims about not understanding what he was saying (“You're never understood” [556]), through objections to his alleged intention to show his advantage and act as their teacher (see Dan’s exclamation “Ziv, you won't be a teacher” [678], and one girl’s complaint to the teacher/researcher: “He just… he talks to me like I'm his [little] girl!” [704]). Ziv reciprocated with explicit reinforcement for the story of his superiority (see his utterance directed at one of the girls: “I’m smarter than you, Idit” [471]). With this mutually aggravated subjectifying ping-pong going on and on, and with the identity-building activity high on everybody’s agenda, Ziv evidently stood little chance to play the role of the leader.

The analysis opened the teacher’s eyes to these “identity struggles” and made her aware of her own central role in the plot. In hindsight, she expressed her regret:

> [T]he conundrum of the children’s tendency to learn from a less competent classmate ... seems to have been solved: the student who could [deal with] the problem was denied the identity of discourse leader… I am [now] able to see things of which, in real time, I was [unaware]. Above all, I realized that my role in the students’ learning was more harmful than helpful. [I] took part in [constructing Ziv’s identity] just like anybody else in this classroom. In fact, my role in this process was probably most central .... It is therefore even more regrettable that I acted the way I did, constructing students’ identities unreflectively, rarely giving my [utterances] a second thought.

Were this insight gained in real time, the teacher would have probably curbed this subjectifying discourse. If the latter did not happen, it was mainly because she clearly remained oblivious to the fact that while trying to advance the mathematizing and repeatedly encouraging Ziv to share his solutions with the classmates, she was also constructing the boy’s first- and third-person identities. She saw herself as preoccupied exclusively with the mathematizing discourse, whereas the students perceived her as performing the task of telling them who they were, and thus as engaged in subjectifying discourse. These two differing visions and the resulting discursive gap are summarized in Table 6.

Table 6
Discourses and routines in Example III

	Teacher (performer)	Students (interpreters)
discourse	mathematizing	subjectifying
task	scaffolding students’ problem solving “by proxy”	building Ziv’s (and other students’) identity
procedure	inviting Ziv to present his solutions, evaluating Ziv’s understanding	exhorting the class to listen to Ziv,

Example IV: The danger of modelling a discourse other than intended

The last example has shown how a gap between the teacher’s own and her students’ perception of her discourse may result in the teacher’s involuntary participation in a harmful subjectifying activity. In the next example, we will see how a similar discursive gap can lead to the teacher’s unconscious support for a wrong type of mathematizing.

While saying “the wrong type of mathematizing” I mean mathematical discourse different in its character from the one the teacher herself intended. Thus, for instance, the teacher may believe she is trying to usher her students to *explorative* mathematizing while, in fact, the way she teaches supports *ritualistic* participation. Indeed, most teachers are likely to wish their students to see themselves as engaged in mathematical *explorations*, that is, in the activity of telling potentially useful stories about mathematical objects. As it often happens, however, the teachers' own way of acting may push their students toward rituals, that is, can make the learners believe their task is merely to show a mastery of mathematical procedures. In this later case, they feel exempted from worrying about the question of what the outcomes of their performances may be good for.

These differing views of the purpose of mathematizing are rarely introduced to the students in the direct manner. Rather, they are signaled by the teacher's discursive moves, especially those finest ones, which are also least noticeable. Among the most effective shapers of the students' interpretations is the teacher's language. Let me illustrate this claim with the example presented in Table 7, in which the teacher who participated in a recent study on teaching algebra in high school (Adler & Sfard, 2018) introduces his class to the process of solving the quadratic equation $(\mathrm{x}-2)(\mathrm{x}+2)=0$.

Table 7
Example IV: Solving $(x-2)(x+2)=0$

#	Speaker	What was said	What was done
1	Teacher:	We want to solve for x. What is our x equal to?	Writes: $(x-2)(x+2)=0$
2	Learners:	……	The learners remain silent
3	Teacher:	We are saying any of these brackets is equal to 0.	
4	Teacher:	So we are saying $x-2$ is equal to 0… OR… $x+2$ is equal to 0	While saying this, I would be writing on the board: "$x-2=0$ or $x+2=0$"
5	Teacher:	And then we transpose them. x is equal to?	
6	Learners:	2… or x is equal to -2	As the learners are saying this, the teacher writes on the board: "$x=2$ or $x=-2$"

Let us scrutinize the teacher's utterances for the objects he is talking about. Note, in particular, that the sentences "We want to solve for x" ([1]), "We are saying any of these brackets is equal to 0" ([3]), "And then we transpose them" ([5]) speak about people's actions (*solve*, *transpose*) with symbols (*x*, *brackets*). Within this context, it is justified to claim that also numerals such as '2' and propositions such as 'x=2' are considered as mere symbols, standing for nothing but themselves. This way of speaking supports ritualization, if only because of the fact that the result of symbolic manipulations seems to be of no further use and thus the performance is the only thing that counts.

To create a proper opportunity for the kind of learning that the teacher believed himself to be promoting, he should have exposed the students to explorative discourse. He would have done better if he reduced talking in terms of symbolic operations and spoke as much as possible in terms of mathematical objects, such as numbers or functions.[5] Thus, in utterance [1], instead of talking about "solving for x", he could have asked about the relevant relations between numbers: "What are the numbers x that, if substituted for x will make the product

[5] The difference between symbols and the mathematical objects is that the objects may remain the same while symbols change. Thus, the number two remains the same whether we refer to it with the symbol '2' (Arabic numeral) or II (Roman numeral), or $^{16}/_{8}$.

of x+2 and x-2 equal to 0?" Alternatively, he could have inquired about a property of a function: "For which numbers x the value of the function y=(x+2)(x-2) is equal to 0?" Utterance [3] that speaks about brackets might have been replaced with a proposition on numbers: "Any of the numbers x+2 and x-2 must be equal to 0". Finally, rather than using the cryptic verb "transpose", implying a physical action, such as rearranging symbols, he could have said, "We subtract 2 from [the numbers/functions on] both sides of the equation". The common feature of all these replacements is that they define the task by specifying the required properties of the outcome. Clearly, this stress on the product signals the legitimacy of *any* procedure that would lead to the required result and as such, ushers the problem solver into explorative discourse.

Many other properties of teachers' discursive actions are likely to encourage students' ritualistic participation[6], but in the present context, I chose to focus on those of them that hide in moves so tiny as to being imperceptible either to the students or to the teacher himself. The differences between the routines of the explorative discourse the teacher saw himself as performing and those of the ritualized discourse his students were likely to perceive are summarized in Table 8.

Table 8
Discourses and routines in Example IV

	The teacher performs	The students see
discourse	explorative mathematizing	ritualized mathematizing
task	demonstrate how to attain mathematical outcomes	demonstrate how to perform mathematical procedures
procedure	discuss the required outcome and perform a number of procedures that lead to this outcome	perform a single procedure repeatedly, giving tips for remembering how it should be done

Summary and conclusions: Why teachers should remain alert to the possibility of communicational gaps

Both examples in this section make a strong case for the teacher's awareness of the possibility of a gap between what she thinks she is doing and what her students actually see. This awareness is important because such gaps may mean that what her students learn is not what she tried to teach them. More specifically, the teacher may find herself collaborating in shaping unwanted, potentially harmful identities, while also introducing the students to mathematical discourse she herself does not appreciate. While in the classroom, therefore, the teacher must keep in mind that any of her moves may be read by the learners as saying something about themselves, if only implicitly; and she has to remember that when it comes to the question of what kind of mathematics the learner experiences, the answer is not so much in general didactic principles or even in detailed lesson plans, as in the finest details of the implementation (Sfard, 2018, p. 124).

[6] For instance, the learner's ideas about the source of mathematical narratives depend, to considerable extent, on *what* the teachers say, and to an even greater extent, on *how* they say it. Thus, the teacher who frequently appeals to the students' memory, who accepts his role as the ultimate judge of correctness and who rarely has recourse to a careful deductive derivation is likely to give rise to the students' conviction about an arbitrary nature of mathematical discourse and of its products.

Discursive gaps as the researcher's opportunities for learning about learning

Whereas both teachers and students have good reasons to be apprehensive of discursive gaps, researchers are more likely to see those gaps as gates to hidden treasures. As could already been understood from the first two examples, valuable insights about learning can be gained from close analyses of the nature of different discursive gaps and of the circumstances that occasion their appearance. In this section, I look at yet another case, in which the occurrence of a gap becomes an opportunity for learning about ways in which people match task-situations with discourses.

Example V: Opportunity to learn about student's ways to choose precedent

The example to be presented now may help researchers in identifying those aspects of task-situations that can be held responsible for students' choices of discourses in which to react to given task-situations. Some relevant insights could already be gained from Example I, where the learner was primed by the formulation of the problem, and more specifically, by words and symbols such as 'times', 'of' or multiplication sign. The new example will show again that two task-situations considered by one person as defining the same task may be seen by another as calling for different routines. This time, however, with the wording of the task-generating question remaining constant, the role of precedent-indicators will be played by contextual factors.

The data to be considered now come from a study conducted in two 7^{th}-grade classes, of 36 students each. The students were presented with the mathematical problem: "Four children shared 14 balloons. How many balloons did every child get?" The two classes could be considered as indistinguishable in terms of the history of their mathematical learning and their achievement, and the only difference between them was that one was asked to solve the Balloons problem during mathematics lesson and the other – during a language lesson. The results can be seen in Table 9.

Table 9
Example V: Students' responses to the Balloons task

Response	Frequency	
	Mathematics lesson (N=36)	Language lesson (N=36)
"3.5"	46%	14%
"The children got 4 and two others got 3 balloons" "Each child got 3 balloons and 2 were left"	50%	80%
NA	4%	6%

As can be seen, the results obtained in the two classes are quite different. During mathematics lesson, almost half of the students responded with the non-integer number 3.5 that could not possibly constitute an answer to the question of the number of balloons. These participants clearly identified the task as a "word problem", the type of problem frequently encountered by every mathematics learner. The procedure they used was the one they often used in this context: finding and implementing the arithmetic operation that seemed to fit the question. In the present case, the division was probably chosen because of the word "sharing" appearing in the statement of the problem. In the other class, this improbable response was given by the mere 14% of the students. The majority of answers seemed to indicate that here, just like in Example I, the children saw it as their task to perform the everyday routine of fair sharing that they often had to perform in their everyday life. Thus, whereas in Example I the difference in the choice of discourse and, in result, in the solution routine stemmed from lexical differences, in this example the decisive factor was the context in which the question was stated. For a summary of this analysis see Table 10.

Table 10
Discourses and routines in Example V

	In mathematics lesson	In language lesson
discourse	everyday	of school mathematics
task	sharing the balloons fairly between children	perform a learned operation that fits the situation
procedure	1. Give a balloon to each child 2. Repeat as long as you can	1. Find the most appropriate operation (“share” → division) 2. Perform the operation

To sum the insight that can be gained from this example, our ability to act in most situations in which we find ourselves stems from our tendency to automatically associate each such situation with a certain discourse and with its routines. What prompts these association are such characteristics of the situation as the physical components of the given space (e.g., a typical classroom arrangement) or the identity of the individuals who populate the scene (e.g., mathematics teacher). The very exposure to these identifiers may suffice to push us into the discourse we encountered under the same or similar circumstances in the past. In Example V, the association with mathematical discourse learned at school was brought by the students’ awareness of their being in mathematics lesson, maybe even by the very presence of the mathematics teacher. If the language lesson did not lead to a similar choice, it was simply because mathematical discourse had never been used in this context.

Example VI: Opportunity for replacing the “deficit model”

The example that follows shows how the researcher’s unawareness of a discursive gap between her and participants of her study may stymie her ability to tell a truly useful story of the phenomena she tries to fathom.

Let us consider the conversation between 4-year old Roni, 4 years and 7 months old Eynat, and Roni’s mother, as presented in Table 11. The excerpt is taken from a study on children’s numerical thinking conducted years ago by Roni’s mother, who was also the beginning researcher, and myself (Sfard & Lavie, 2005). The conversation was held in Hebrew (in its English version, presented here, we tried to preserve idiosyncrasies of the children’s language). At the time of our investigations, Roni and Eynat were already quite proficient in counting and were routinely answering the “How many?” question without a glitch. The episode began when the mother presented the girls with two identical opaque boxes. Even though the girls they could not see the contents, they knew they boxes contained marbles. On the face of it, nothing new can be learned from this example. After all, the first thing one usually learns from books and articles about early numerical thinking is that “children who know how to count may not use counting to compare sets with respect to number” (Nunes & Bryant, 1996, p. 35). Yet, at a closer look, some of Roni’s and Eynat’s actions did appear puzzling. If a person was listening to the conversation without seeing the boxes, she would have been likely to conclude that the children implemented the task properly: they gave an agreed answer and knew how to justify it in a logical way (see utterances [5], [7], [9]). But for those who could actually see what was happening, the girls' decisive responses were difficult to account for. Indeed, why did the children choose a particular box? Why did they experience no difficulty in making a joint decision? Why, in the end, were they able to respond in a seeming reasonable way to the request for substantiation, even though there was no basis for the claims they made about the size of the collections?

Table 11
Example VI: Where are there more marbles?

#	Speaker	What is said	What is done
3a	Mother:	Right, there are marbles in the boxes. I want you to tell me in which box there are more marbles	
3b	Eynat:		Points to the box which is closer to her
3c	Roni:		Points to the same box.
4	Mother:	In this one? How do you know?	
5	Roni:	Because this is the biggest than this one. It is the most.	
6	Mother:	Eynat, how do you know?	
7	Eynat:	Because… cause it is more huge than that.	
8	Mother:	Yes? Roni, what do you say?	
9	Roni:	That this is also more huge than this.	

After long deliberations and a scrutiny of children's actions in this and similar episodes, we concluded that it was the language used in the description of the case that produced our puzzlement. Indeed, while stating that children do "not use counting *to compare sets with respect to number*" (emphasis added), the researchers attribute to children their own interpretation of the question "Where are there more marbles?" If so, there is little wonder they view children's actions as suffering from a certain deficit: the girls did have the necessary skill but they were unable or unwilling to use it the way they, the researchers, would have used it themselves in the same task-situation. The story of the deficit loses grounds, however, when one realizes that Roni and Eynat did not necessarily interpret the question "Where are there more?" as requiring quantitative comparison. Indeed, having freed oneself from the assumption, one realizes that, perhaps, the children simply tried to choose the box that they *preferred*. As implied by previous studies (see e.g., Walkerdine, 1988), rather than interpreting the word 'more' as referring to quantitative advantage, they were likely to understand it as referring to whatever could count as better, for one reason or another. In sum, we understood that there was a gap between the children's and grownups' visons of the task, and thus between their respective discourses and routines. These differences are summarized in Table 12.

Table 12
Discourses and routines in Example VI

	Interviewer	4 year old children
discourse	quantitative, numerical	of choosing for oneself
task	identify the box that has more marbles	choose (together?) the box you (both?) prefer
procedure	1. Count marbles in each box 2. Compare the last number words obtained in B	Point to, or take, the one you prefer (possibly: trying to agree with your friend)

The insight gained in this event had a lasting impact on our later work. From now on, we have been avoiding telling stories on what children did not do and, instead, have been documenting what they actually did. The sentence "children who know how to count may not use counting to compare sets with respect to number" has now been reformulated in our reports as "Children who know how to count, when asked 'Where is there more?', are likely to make a choice without counting".

The importance of the lesson that can be learned from this example by both teachers and researchers cannot be overestimated. When students seem to err, we tend to assume that the error is due to their insufficient mastery of procedures. It occurs to us only rarely, if ever, that the apparent mistake may result from a difference between the task the learners try to perform and the one intended by the task-setter. Yet, what we saw in this example alerts us to the fact that when a routine develops, transformations in the students' vision of the task may be at least as significant as the gradual increase in these students' mastery of procedures. To do their job properly, those who teach and those who investigate learning must bracket their own mathematical discourse. They should always try to present the one's performance as it was seen by the performer herself. This is the only way to disrupt the long tradition of portraying the learning of mathematics as a process of overcoming lingering deficit. To begin picturing learning as a series of creative advancements towards an ever greater complexity, the researcher must always remember that the journey to full-fledged participation in historically established mathematical discourse involves traversing multiple, possibly invisible discursive gaps.

Summary and conclusions: Wariness of communicational gaps as a protection against deficit model of learning

The two latest examples as well as some of the previous ones make it abundantly clear that researchers should embrace discursive gaps as opportunities for their own learning rather than just problems to solve. The first of these examples has shown how a recognized discursive gap becomes a window to inner workings of the process of learning. Through this window we had a close-up at the way people choose precedents to task-situations, and what we saw shed light on the phenomenon known as *situativity of learning* (Brown et al., 1989; Greeno, 1997; Lave, 1988). The second example brought a message about some hitherto unrecognized pitfalls, in which we often fall as researchers. Here, we saw how our own mathematical discourse may blind us to critically important aspects of children's activity, making us oblivious to the mechanisms of discourse development. It warns the researchers against relying on their own mathematical discourse while trying to make sense of what children are doing.

Coda

In this talk, I joined Wittgenstein in his "battle against the bewitchment of our intelligence by means of our language" (Wittgenstein, 1953/1967, p. 47). Diverse ways in which language may lead us astray have been illustrated with multiple examples. These examples were also used to show how important it is that all the parties to processes of teaching and learning, whether participants or observers, are always alert to the possibility of discursive gaps. The examples illustrated the claim that some of these gaps are inevitable. I argued that these ineluctable discursive discontinuities should be embraced as opportunities for learning. Those gaps that do little more than jeopardize learning – and my examples imply that these are not any less frequent than the useful ones – can and should be prevented. In all the cases, however, the devil hides in the tiniest details of interpersonal communication and our first task is to learn how to make the gaps visible. Unknown to the teacher, her basic communicational routines may constitute invisible crevices through which the prejudice enters the conversation on mathematical objects.

It would be naïve to think that the uneasy task of detecting and preventing or utilizing discursive pitfalls could be implemented without a deliberate effort. Echoing Michael Reddy, successful exchange "cannot happen spontaneously or of its own accord" (Reddy, 1979, p. 296). Remembering that "[h]uman communication will almost always go astray unless real energy is expended." (p. 295), we need to invest as much energy as possible in minding even those discursive gaps that at the moment remain invisible.

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